



# Global Carbon Finance

**A quantitative modelling framework to explore scenarios of the Global Deal on Climate Change**

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## **Abstract.**

**The Kyoto Protocol is set to expire in 2012 and there is now urgency to reach a new international agreement to replace it if the world is to avoid dangerous climate change. Negotiations are under way, and an international framework to tackle global warming is expected to be agreed at the end of 2009 in Copenhagen under the UNFCCC.**

**To achieve this, the world will need excellent modelling capabilities to estimate the costs of mitigation. Currently there are a number of numerical carbon finance models, such as GLOCAF and FAIR. However, such models require significant effort to develop and maintain, and most governments do not own their own models. One problem is that, as we enter the final negotiation stages, governments will need to rely and trust second hand estimates of mitigation costs from a small number of models held in a few developed countries. This information asymmetry may lead to lack of trust and ultimately adversely affect the negotiations. This paper attempts to address this issue.**

**The purpose of this paper is to help Governments and research institutions kick-start and accelerate the development of their in-house modelling capabilities to analyse the shape of the Global Deal on Climate Change. More concretely, the model offers the opportunity to readily obtain estimates of costs and financial flows, as well as other quantities of interest. The paper presents a set of formulae that could be readily used to set up a simple spreadsheet model to analyse the climate change negotiations. Moreover, the paper also describes the underlying modelling framework, which can be used as a platform to design and analyse new scenarios and generally to develop more complex models of the global deal on climate change.**

**The validity of the model results is tested against state-of-the-art numerical models (GLOCAF and FAIR), and is found to be in excellent agreement. Moreover, the mathematical formulation presented in this paper was used to develop the UK's Global Carbon Finance model, GLOCAF. Applications of this model have so far been presented to the EU Commission, Number 10 as well as to the UNFCCC conference in Bali. Furthermore, the model is also being used to facilitate technical discussions between the UK, India, Brazil, Mexico, Argentina and Colombia.**

**It is hoped that this model will help to accelerate the process of reaching an international agreement on climate change.**

## Summary of key formulae

For quick reference, the boxes below summarise the key equations from the carbon finance model. These are derived formally and fully explained in the following pages.

### Box 1: Key formulae (free trade)

Equilibrium carbon price:  $p_{eq} = R \cdot Q^\alpha$

#### Global

- Net costs [\$bn]:  $C_G^C = \frac{R}{\alpha + 1} Q^{\alpha+1}$

#### Regional

- Domestic abatement [GtCO<sub>2</sub>e]:  $q_i^d = \left(\frac{R}{\rho_i}\right)^{1/\alpha} \cdot Q$
- Net costs [\$bn]:  $C_i^N = p_{eq} \cdot q_i^r - D + C_i^I$
- Financial flows [\$bn]:  $F_i^F = p_{eq} \cdot q_i^r - D$
- Carbon flows [GtCO<sub>2</sub>e]:  $F_i^C = q_i^r - q_i^d$

### Box 2: Related constants (free trade)

- Domestic credits [\$bn]:  $D = q_i^d \cdot p_{eq}$
- Incremental costs [\$bn]:  $C_i^I = \frac{\rho_i}{1 + \alpha} \left[ \left(\frac{R}{\rho_i}\right)^{1/\alpha} Q \right]^{1+\alpha}$
- Abatement targets [GtCO<sub>2</sub>e]:  $\sum_i q_i^r = Q$

$$R = \left( \frac{1}{\sum_i \left(\frac{1}{\rho_i}\right)^{1/\alpha}} \right)^\alpha$$

## 1. Introduction

Scientific research and political negotiations on climate change have been going on for decades. However in the past couple of years the debate has been gathering momentum thanks to a number of factors. Supported by growing evidence, the scientific community has reached an almost unanimous consensus that climate change is real and is happening at a faster rate than previously predicted. Also, awareness among the general population has grown considerably, with the issue now appearing regularly on the front pages of national newspapers, and the media constantly presenting new scientific evidence.

A series of events have also contributed to putting climate change at the top of the international political agenda. In 2006 the Stern Review provided the first systematic economic assessment of the potential consequences of inaction and the costs of mitigation (Stern, 2006). The 2007 Noble Peace Prize gave further impetus to the political debate. Also business is becoming an active voice; for example, 150 of the biggest companies in the world demanded an international agreement on climate change in November 2007. The UN conference in Bali in December has initiated negotiation talks between the world leaders and set up a Road Map to design the post-Kyoto framework. The G8 summits in both Germany (2007) and Japan (2008) concluded that climate change is a real threat and that the industrialised nations are committed to cutting greenhouse gas emissions significantly in the near to mid-term.

Now countries are entering full negotiation mode, and is expected that by the end of 2009 a political agreement should have been reached in Copenhagen. However, before countries commit to legally binding targets, they will need to understand the financial costs involved. Therefore, excellent modelling capabilities will need to be available to inform decision-makers. The UK's GLOCAF model is one of the tools capable of quantifying the costs and financial flows involved in many negotiation scenarios. An alternative model is MNP's FAIR. One weakness of these models is that they are complex computer simulations that require significant resources to develop and maintain. Moreover, these models can be very data hungry, and to both obtain and update the relevant data can be both time consuming and expensive.

Because of this, currently many developing countries do not have in-house carbon finance models, and instead often rely on data from foreign institutions to design their negotiation strategy. This information asymmetry may affect levels of trust and have a negative impact on the negotiation process.

The purpose of this paper is to address this issue by presenting a mathematical formulation that would allow Governments and research institutions that do not yet possess these capabilities to develop their in-house model to describe and analyse scenarios of the Global Deal on Climate Change.

Starting from greenhouse gas emission targets and marginal costs of abatement, this paper develops a set of simple formulae that describe a global carbon finance market. The formulae are derived in two different ways: by either assuming minimisation of abatement costs or by matching demand and supply of the carbon

credits. Both methods are shown to be equivalent. The model outputs are quantities of interest such as international financial flows and net costs incurred by individual countries.

It is hoped that this mathematical formulation is more transparent and concise, in the sense that it does not require complex numerical simulations. Also, the data requirements are substantially reduced, thus constraining the required investment in terms of time and money.

Section 2 presents the assumptions underlying the model: emission targets and marginal abatement costs. Sections 3, 4 and 5 present the results of the model. In particular, Section 3 discusses the equilibrium carbon price and the domestic abatement. Section 4 looks into the country specific results, while Section 5 presents the global costs. Section 6 concludes and suggests further research. A number of technical appendices explain the formal derivations of the model and illustrate some practical applications that may be used to support the international negotiations as well as the design of the global carbon market.

## **2. Assumptions: global emissions target and marginal abatement costs**

It is of interest to obtain an estimate of the costs that would be incurred by countries as they reduce their greenhouse gas emissions. If an international carbon market is set up, countries will also be interested in the resulting financial flows. In particular, individual countries will need to be confident in their numbers as they join the negotiation table.

In order to obtain an estimate of these quantities we need three pieces of information (Ellerman and Decaux, 1998):

1. the greenhouse gas emissions targets,
2. the marginal costs of abatement, and
3. the mechanism by which the abatement is achieved: here we consider an international carbon market with rational actors<sup>1</sup>.

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<sup>1</sup> By 'rational actors' we mean utility-maximising economic actors as defined in microeconomic theory. This has some limitations, for example in the case of energy efficiency where there may be barriers such as split incentives whereby the landlord does not insulate the property because he will not gain from energy efficiency savings, while the tenants will not benefit from the higher price of the property.

## 2.1 Greenhouse abatement targets

The global level of abatement,  $Q$  (measured in GtCO<sub>2</sub>e), may be internationally agreed and should be informed by the latest science as to avoid dangerous anthropogenic climate change. This global target is then shared among countries according to a burden sharing scheme: each country  $i$  is required to accept a national abatement target  $q_i^{\tau}$  (also measured in GtCO<sub>2</sub>e) so that the global target is met<sup>2</sup>. The national targets should add up to the global target:

$$Q = \sum_i q_i^{\tau} \quad (1)$$

## 2.2 Marginal costs of abatement

The costs of abatement may be described by Marginal Abatement Cost (MAC) curves. These may be disaggregate at regional (national) or sectoral level, examples of which are the steel and cement industries. MAC curves may be obtained in a number of ways; for example, from general equilibrium models, energy models or from an inventory of abatement options, such as energy efficiency or the introduction of green technologies (Kuik et al, 2008).

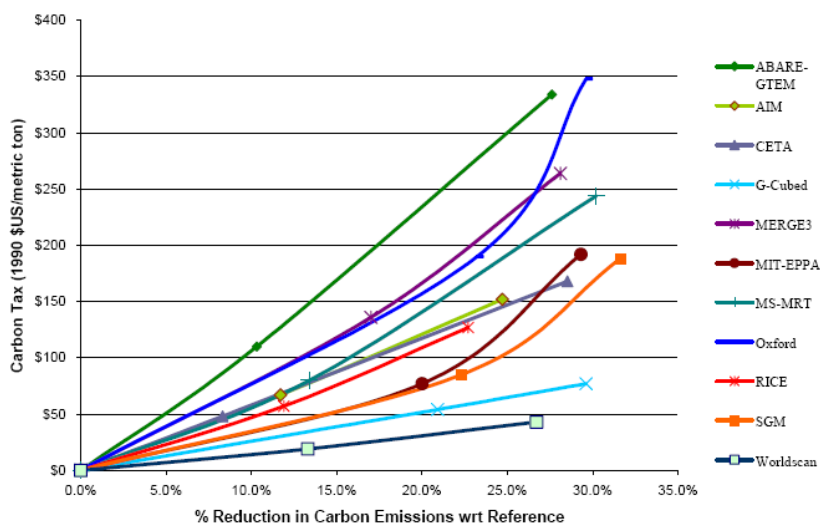
In general the data may be approximated by a function  $p_i=f(q_i)$ , where  $p$  is the marginal abatement cost (or equivalent carbon price) and  $f(q_i)$  is an arbitrary function of abatement,  $q_i$ . The quantity  $p$  measured in US\$/tCO<sub>2</sub>e. For a given country  $i$ , this equation represents the amount of carbon  $q_i$  that will be economically viable to abate at a given carbon price  $p_i$ . GLOCAF uses empirical data to specify the equation. It is found that the following family of functions can accurately approximate the data (see Appendix II):

$$p_i = \rho_i \cdot q_i^{\alpha} \quad (2)$$

The values of the parameters  $\rho_i$  are obtained from the data through standard statistical estimation – e.g. least square fit – and are specific for each MAC curve. (An individual MAC curve could be global, regional or sectoral, depending on the underlying data.) The parameter  $\alpha$ , on the other hand, is common to all curves and is also estimated empirically. See Appendix II for further data considerations.

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<sup>2</sup> In practise, in some cases the targets may be set equal to zero (e.g. developing countries participating in CDM schemes).



**Figure 1: Different models produce substantially different estimates of marginal abatement cost curves. Source: Fischer and Morgenstern, 2005.**

Equations 1 and 2 are very simple but powerful. These models are subject to significant levels of uncertainty: for example, Figure 1 shows how estimates of marginal abatement costs vary significantly depending on the approach used to generate them (Fischer and Morgenstern, 2005; Kuik et al, 2008). The formulation of the present global carbon finance model is such that allows us to use different underlying datasets and explore the sensitivity to them. Moreover, the uncertainty can be incorporated into the model to obtain a probabilistic version: this would allow us to obtain *ranges rather than exact estimates* of the international costs and financial flows. Although turning the deterministic model into a probabilistic one is relatively straightforward, as major constraint is the data availability: currently there are no standard ‘probabilistic MAC curves’, and producing them would require a number of ad-hoc considerations.

Equations 1 and 2 also help us focus on the sources as well as the implications of these underlying uncertainties.

## 2.3 The carbon market

Each country could be responsible for meeting its entire abatement target domestically. However, it is a well known economic fact that the total costs can be considerably reduced if countries are allowed to buy carbon allowances from countries where abatement is cheaper (Stern, 2007). Indeed, the Kyoto Protocol allows for such an international carbon market. Moreover, a number of such markets already exist: the EU’s Emission Trading Scheme is currently the largest, but other are being set up, for example in Australia and the United States.

In Appendix I, carbon trading rules<sup>3</sup> are applied to the above marginal abatement costs and emission abatement targets to derive explicit formulae that describe the international carbon market; these include, for example, the global costs, regional costs, financial flows and can be easily manipulated to deduce further quantities of interest. The formulae can be derived by two different methods: both by applying cost minimisation conditions or by matching demand and supply of carbon credits. The following sections describe these quantities in more detail.

### 3. Key results: equilibrium carbon price ( $p_{eq}$ ) and domestic abatement ( $q_i^d$ )

This section presents the two key results of the carbon market model: the equilibrium carbon price and the level of domestic abatement. These are special quantities because all the other model outputs can be derived from them through algebraic manipulation.

#### 3.1 The equilibrium carbon price

The exchange of carbon credits in the international market would give rise, by matching demand and supply, to an equilibrium carbon price or market clearing price,  $p_{eq}$  (measured in US\$/tCO<sub>2</sub>e). This is the first key result of the model, and is given by:

$$p_{eq} = R \cdot Q^\alpha \quad (3)$$

The constant  $R$  is given by:

$$R = \left( \frac{1}{\sum_i \left( \frac{1}{\rho_i} \right)^{1/\alpha}} \right)^\alpha \quad (4)$$

As can be seen,  $R$  depends only on the marginal abatement costs of individual countries, as measured by the empirical parameters  $\rho_i$  and  $\alpha$ .

Equation 3 offers a number of important insights:

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<sup>3</sup> The carbon trading rules are based on classical economic theory, which assumes perfectly rational behaviour.

1. First of all, it shows that the equilibrium carbon price depends only on the global emissions abatement target,  $Q$ ; in other words it does not matter how this target is allocated among nations. This is consistent with economic theory<sup>4</sup>.
2. It is also interesting to compare Equation 3 with Equation 2: this allows us to see that the equilibrium carbon price corresponds to the global marginal abatement cost of meeting the global target  $Q$ .<sup>5</sup>
3. Another insight is that, since the incremental cost is the integral of the marginal abatement cost (Equation 3), it should follow that the global incremental cost is given by:  $C_G^C = \frac{R}{\alpha + 1} Q^{\alpha+1}$ . This is indeed the case, and will be confirmed below.

### 3.2. The domestic abatement

Under free trade, the amount of abatement that is carried out in each country is independent of the country's abatement target: countries will abate up to the point where the marginal cost of abatement is equal to the carbon price. This is because when abatement costs are higher than the market carbon price, it would be more convenient to buy credits from the market. On the other hand, if the national target is exceeded, the country will be able to sell carbon credits at a higher price than they spent on abatement to create them.

The second key result of the model is an explicit formula for the level of domestic abatement,  $q_i^d$  (measured in GtCO<sub>2</sub>e), that is economically optimal to abate in each country  $i$ . This is done by rearranging Equation 2 in terms of  $q_i$ , and replacing the carbon price with Equation 3:

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<sup>4</sup> It is important to note that this statement is valid only in the case of free trade. Restrictions to trade would result in higher costs.

<sup>5</sup> As a matter of fact, the parameter  $R$  can either be obtained from the  $\rho_i$  through Equation 4, or it can be estimated empirically directly from the global marginal abatement cost curve (Equation 3). The values obtained from the two approaches would differ slightly. This is because of the way the statistical errors enter in the parameter estimations: if the  $R$  is estimated directly from the global cost curve, the associated error will result from the standard least squares estimation method. On the other hand, if the  $R$  is obtained from the above formula, its value would incorporate all the errors associated with the individual  $\rho_i$ . In some cases these errors may cancel out, however in other cases these may add up. Therefore attention should be paid to these issues during the estimation of the parameters. In general, if we are interested in global costs, it may be more accurate to estimate  $R$  from the global cost curve. On the other hand, if we are interested in regional costs and flows, it would be necessary to obtain  $R$  from the regional  $\rho_i$ , as this would ensure that total financial flows add up to zero.

$$q_i^d = \left( \frac{R}{\rho_i} \right)^{1/\alpha} \cdot Q \quad (5)$$

This corresponds to the level of abatement that each country will undertake domestically under free trade. In other words, under a completely free market each country will reduce emissions domestically until the domestic marginal abatement cost matches the global carbon price. Each country will then be able to buy or sell carbon credits in the international carbon market to meet its agreed abatement target. Countries with high domestic marginal abatement costs will be able to purchase carbon credits at lower costs on the market, while countries with cheap abatement costs will be able to abate more than their target and then sell the credits at a higher price, thus making a profit.

Equation 5 shows explicitly how the level of domestic abatement is independent of the country's abatement target (i.e. the abatement target,  $q_i^t$ , does not appear in the formula). On the other hand, the domestic abatement is a fixed fraction of the global abatement target. This proportion will be larger in those countries where the abatement costs are smaller (i.e. smaller  $\rho_i$ ). In other words, countries with cheaper abatement costs will abate more domestically, as we would expect.

The next two sections present a number of regional and global quantities of interest: these are obtained through algebraic manipulations based on the key formulae presented above.

## 4. Regional costs and flows

Negotiating a burden sharing agreement is a phenomenal task due to the large number of parties involved, their different geopolitical weights and the widely diverging national interests and circumstances. Moreover, different levels of economic development and historic emissions, as well as ethical issues add to the complexity of the problem. Before any country commits to a mandatory emissions target, it will need to assess and accept the responsibilities that come with it; for example the risk of having to pay a fine if the target is not met. Therefore, countries will need estimates of the incremental costs and financial flows involved.

This and the next sections describe how these quantities can be derived. In essence, Equations 3, 4 and 5 describe the global carbon finance model: by manipulating them it is possible to define and calculate simple formulae for country specific quantities such as incremental costs or financial flows. This section derives formulae for the most frequently used financial quantities. It is important to note, however, that the list below is by no means exhaustive, and more quantities can be defined if necessary.

## 4.1 Carbon flows

As an initial example, define the regional carbon flows,  $F_i^C$  (measured in GtCO<sub>2</sub>e) as the difference between the country's abatement target,  $q_i^r$ , and the level of domestic abatement,  $q_i^d$ .

$$F_i^C = q_i^r - q_i^d = q_i^r - \left(\frac{R}{\rho_i}\right)^{1/\alpha} \cdot Q \quad (6)$$

Here positive flows indicate that the country is buyer of carbon credits, and vice versa.

## 4.2 Financial flows

As discussed above, carbon credits will be traded at the equilibrium carbon price,  $p_{eq}$ . Therefore, it is straightforward to obtain the financial flows,  $F_i^F$  (measured in US\$bn), by multiplying the carbon credit flows by the equilibrium carbon price:

$$F_i^F = p_{eq} \cdot F_i^C = \left[ q_i^r - \left(\frac{R}{\rho_i}\right)^{1/\alpha} \cdot Q \right] \cdot R \cdot Q^\alpha \quad (7)$$

We can obtain a more intuitive formula for the financial flows by defining the domestic credits as:  $D_i = q_i^d \cdot p_{eq}$ . This is the market value of the credits generated through domestic abatement. We are now in a position to rewrite the financial flows as:

$$F_i^F = p_{eq} \cdot q_i^r - D_i \quad (7a)$$

In words, this is saying that the financial flows are given by the difference in carbon credits that would be needed to meet the national target and the credits generated domestically. The difference between the two would be either sold to or bought from the market.

### 4.3 Incremental regional costs

Each country  $i$  will incur incremental costs,  $C_i^I$  (measured in US\$bn), due to the amount of carbon abated domestically. This can be obtained by taking the integral of the marginal abatement curve up to the level of domestic abatement, which is given by Equation 5:

$$C_i^I = \frac{\rho_i}{1+\alpha} \left( \left( \frac{R}{\rho_i} \right)^{1/\alpha} \cdot Q \right)^{1+\alpha} \quad (8)$$

Equation 8 shows that the incremental costs for each country are independent of the abatement targets. Since it was already shown that the level of domestic abatement, Equation 5, is also independent of the abatement target, this is not surprising.

### 4.4 Net regional costs

The net costs of each country,  $C_i^N$  (measured in US\$bn), will be the sum of the financial flows and incremental costs of the country. They are given by:

$$C_i^N = \frac{\rho_i}{1+\alpha} \left( \left( \frac{R}{\rho_i} \right)^{1/\alpha} \cdot Q \right)^{1+\alpha} + \left( q_i^r - \left( \frac{R}{\rho_i} \right)^{1/\alpha} Q \right) \cdot R \cdot Q^\alpha \quad (9)$$

Equation 9 can be rewritten in a more intuitive form by substituting in the formulae for the domestic credits,  $D_i$ , incremental costs (Equation 8):

$$C_i^N = p_{eq} \cdot q_i^r - D_i + C_i^I \quad (9a)$$

In words, Equation 9a is saying that the net costs are given by the incremental costs and the financial flows. The above formula shows that the net costs depend on the national abatement target, as well as on the national and global marginal abatement costs and global abatement target.

### 4.5 Net regional costs in the absence of carbon trading

For comparison, it is useful to determine the net costs,  $K_i$ , that each country would experience in the absence of an international carbon market:

$$K_i = \frac{\rho_i}{\alpha + 1} (q_i^{\tau})^{\alpha+1} \quad (10)$$

This is obtained by replacing the term within brackets in Equation 8 by  $q_i^{\tau}$ .

## 5. Global costs and flows

The formulae presented in the previous section describe costs and flows that apply to individual regions. The section presents a number of global variables.

### 5.1 Net global costs

The net global costs of abatement are given by the sum of all the country specific incremental costs (Equation 8):  $C_G^C = \sum_i C_i^C = \sum_i \frac{\rho_i}{1+\alpha} \left( \left( \frac{R}{\rho_i} \right)^{1/\alpha} Q \right)^{1+\alpha}$ . This is measured in US\$bn. Simplifying and substituting for  $R$ , it follows that the global cost of abatement is given by the simple formula below:

$$C_G^C = \frac{R}{\alpha + 1} Q^{\alpha+1} \quad (11)$$

This is a simple yet very powerful formula. Once the value of the parameters  $R$  and  $\alpha$  has been determined empirically, it becomes straightforward to obtain the global costs of meeting the global abatement target,  $Q$ . For example, we can draw any emission pathway and, given a BAU trajectory, automatically get an estimate of the associated costs by calculating the global abatement target,  $Q$ . Moreover, Equation 11 allows us to gain more insight into the problem. For example, it shows that the incremental costs increase nonlinearly with abatement. This means that a small increase in the level of abatement can lead to a large increase in the resulting costs.

### 5.2 Net global costs in the absence of trade

For comparison, the global costs (again measured in US\$bn) in the absence of trading may be calculated in the same way. This results in the following formula:

$$K_G = \sum_i K_i = \frac{1}{\alpha+1} \sum_i \rho_i (q_i^r)^{\alpha+1} \quad (12)$$

It is interesting to note that in this case the costs are a function of the abatement target, as this determines the amount of abatement done domestically.

## **6. Conclusions and suggestions for applications and further research**

We developed a new mathematical model to describe a global carbon trading market.

The model is transparent, and offers new insights. For example, we showed that it may be wise to have more open discussions on the underlying data, specifically:

1. business as usual projections,
2. global abatement targets and
3. marginal abatement costs

before determining national targets or the characteristics of the trading system.

The model results were compared with state-of-the-art numerical models, showing that the model is accurate.

Perhaps the most powerful property of this model is that it produces simple formulae instead of numerical outputs. These formulae could then be used in other applications.

## Acknowledgements

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## References

**Ellerman AD and Decaux A** (1998) Analysis of Post-Kyoto CO<sub>2</sub> emissions trading using Marginal Abatement Curves. MIT Joint Program on the Science and Policy of Global Change. <http://hdl.handle.net/1721.1/3608>

**Fischer C and Mongenstern RD** (2005) Carbon abatement costs. Why the wide range of estimates? *Discussions paper. Resources for the future.*

**Gallo F, Contucci P, Coutts A and Gallo IA** (2008) Tackling climate change through energy efficiency: mathematical models to offer evidence-based recommendations for policy makers. *Quantity and Quality. Springer.*

**Kuik O, Brander L and Tol RSJ** (2008) Marginal abatement costs of carbon dioxide emissions: a meta-analysis. *ESRI Working paper No. 248.* [www.esri.ie](http://www.esri.ie)

**Larsky M** (2003) The Economic cost of reducing emissions of greenhouse gases: a survey of economic models. Washington DC: Congressional Budget Office.

**Stern N** (2007) The economics of climate change. The Stern review. Cambridge University Press.

**Weyant JP and Hill JN** (1999) Introduction and Overview. *Energy Journal Special Issue: vii-xiv.*

## **Appendix I – Derivation of formulae**

This section formally derives the formulae presented in the main text. The derivations presented here assume free trade, but these can be generalised to include policy restriction to the trade of carbon credits, for example constraints on supplementarity.

Assume the existence of an international carbon market, as well as country specific abatement targets and abatement costs. Classical economic theory suggests that countries will act in order to minimise the costs incurred by their economies. They can choose how much of their target to meet through domestic reductions and how much by buying carbon credits from the international market: economic theory suggests that demand should match supply. Based on this, this section presents two alternative derivations of the carbon trading formulae:

1. one by matching demand and supply and
2. the other by minimising costs.

The two derivations produce the same results: this confirms their validity.

### **Matching demand and supply**

This Section provides a simple proof of the equilibrium price formula by matching supply and demand.

The total abatement target (see Equation 1) is:

$$Q = \sum_i q_i^\tau \tag{IV.1}$$

where  $q_i^\tau$  are the regional targets<sup>6</sup>. The marginal abatement cost curve is given by (see Equation 2):

$$p_i = \rho_i \cdot q_i^\alpha \tag{IV.2}$$

which can be rearranged as:  $q_i = \left( \frac{p_i}{\rho_i} \right)^{1/\alpha}$ .

The carbon credit flows are defined as:

$$\phi_i = q_i^\tau - q_i^d \tag{IV.3}$$

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<sup>6</sup> The abatement targets, measured in GtCO<sub>2</sub>e, are generally a user input. However, it is possible to treat them as a dependent variable. This would allow us to ‘reverse engineer’ a solution: for example it is possible to set a target in terms of financial flows or incremental costs and obtain the corresponding GHG abatement target.

where  $q_i^d$  is the level of domestic abatement, which is one of the quantities that we want to determine.

At equilibrium the sum of the flows must be zero:

$$\sum_i \phi_i = 0 \quad (\text{IV.4})$$

In other words, supply matches demand.

Substituting Equations IV.1 and IV.2 into IV.4, we obtain:

$$0 = \sum_i q_i^\tau - q_i \Rightarrow \sum_i q_i^\tau = \sum_i \left( \frac{p_i}{\rho_i} \right)^{1/\alpha} \quad (\text{IV.5})$$

Economic theory says that at market equilibrium demand will match supply and an equilibrium price will be established. We can replace all the  $p_i$  with an equilibrium price  $p_{eq}$  common to all regions. This allows us to extract  $p_{eq}$  from the summation term:

$$\sum_i q_i^\tau = (p_{eq})^{1/\alpha} \sum_i \left( \frac{1}{\rho_i} \right)^{1/\alpha} \quad (\text{IV.6})$$

Now we can solve for the equilibrium price (see Equation 4):

$$p_{eq} = \left( \frac{\sum_i q_i^\tau}{\sum_i \left( \frac{1}{\rho_i} \right)^{1/\alpha}} \right)^\alpha = \left( \frac{Q}{\sum_i \left( \frac{1}{\rho_i} \right)^{1/\alpha}} \right)^\alpha \quad (\text{IV.7})$$

Define

$$R = \left( \frac{1}{\sum_i \left( \frac{1}{\rho_i} \right)^{1/\alpha}} \right)^\alpha \quad (\text{IV.8})$$

So, as shown in Equation 3, the equilibrium carbon price is given by:

$$p_{eq} = R \cdot Q^\alpha \quad (\text{IV.9})$$

This allows us to find the actual domestic abatement under free trade (see Equation 5):

$$q_i^d = \left( \frac{R}{\rho_i} \right)^{1/\alpha} Q \quad (\text{IV.10})$$

## Minimising costs

This subsection shows that the same results can be obtained by assuming that countries trade in order to minimise their abatement costs.

As before, the marginal abatement costs are given by the function  $p_i = \rho_i \cdot q_i^\alpha$  (Equation IV.2). We can calculate the abatement costs incurred by individual regions under no trading (i.e. the incremental costs):

$$c_i = \int_0^{q_i^d} (\rho_i \cdot q_i^\alpha) dq = \frac{\rho_i}{1+\alpha} (q_i^d)^{1+\alpha} \quad (\text{IV.11})$$

If we sum over all countries (assume there are  $N$  countries), we can obtain the global costs:

$$C = \sum_i^N c_i \quad (\text{IV.12})$$

We now want to minimise global costs subject to the constraint imposed by the global abatement target,  $Q$ :

$$\begin{aligned} \text{Min}_{q_i^d} : C \\ \text{s.t.} : \sum_i^N q_i^d = Q \end{aligned} \quad (\text{IV.12})$$

Where  $q_i^d$  are the amounts to be abated domestically by individual countries under free trade: countries will abate until the marginal cost in their MAC curve equals the equilibrium carbon price.

The Lagrangian in this case is given by:

$$L = \sum_i^N \frac{\rho_i}{1+\alpha} (q_i^d)^{1+\alpha} - \lambda \left( \sum_i^N q_i^d - Q \right) \quad (\text{IV.13})$$

The equilibrium conditions are given by:

$$\begin{cases} \frac{\partial L}{\partial q_i} = \rho_i \cdot (q_i^d)^\alpha - \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = \sum_i^N q_i^d - Q = 0 \end{cases} \quad (\text{IV.14})$$

The second order derivative is positive, confirming that this is indeed a minimum (i.e. minimum cost):

$$\frac{\partial^2 L}{\partial q_i^2} = \frac{\rho_i}{\alpha - 1} \cdot q_i^{\alpha-1} > 0 \quad (\text{IV.15})$$

The solutions to the system IV.14 is given by:

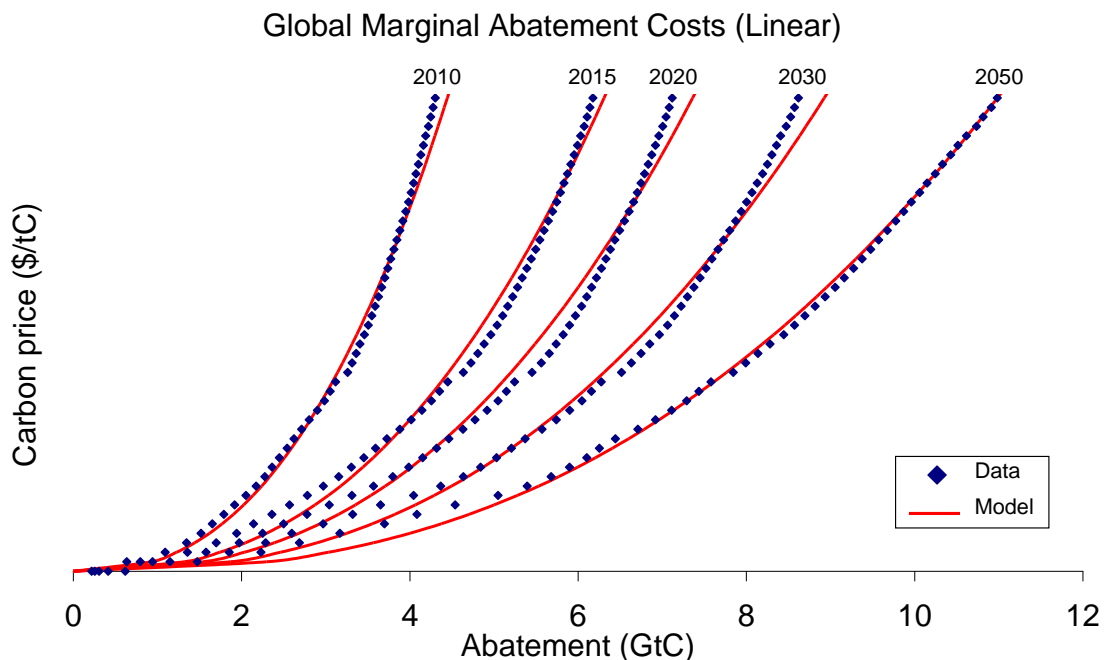
$$\left\{ \begin{array}{l} \lambda = \left( \frac{Q}{\sum_i \left( \frac{1}{\rho_i} \right)^{1/\alpha}} \right)^\alpha \\ q_i = \left( \frac{\lambda}{\rho_i} \right)^{1/\alpha} \end{array} \right. \quad (\text{IV.16})$$

Where  $\lambda$  corresponds to the equilibrium carbon price  $p_{eq}$ . Since Solution IV.16 is the same as solutions IV.7 and IV.10, we can conclude that the carbon trading model obeys both costs minimisation and matches supply and demand. Moreover, since the solutions were obtained from two independent methods, we can infer that they are correct.

## Appendix II – Data considerations

It was explained earlier empirical data is needed to estimate the MAC functions (Equations 2). Here we use POLES data for fossil fuel emissions, GCOMAP data for deforestation and MNP data for non-CO<sub>2</sub> greenhouse gases (Please refer to the acknowledgements sections for more details on the sources of the underlying data). Together, these amount to about 98% of greenhouse gas emissions. These sets of data, can be parameterised by  $\alpha=2.5$  (individual regions would have independent  $\rho_i$ ). Figure 1 shows the fit of the function to the data. The diagram shows global marginal abatement costs for a number of different years. It can be seen that the fit is accurate and very satisfactory. The function is generally accurate to within 5%, and often better than this. Although the examples shown below apply to global abatement costs only, Equation 2 is very general, and can be specified to specific sectors as well: for example, the model presented in this paper could be used to describe a global carbon market that includes the power sector only.

Considering that different approaches result in marginal abatement cost estimates that vary by a factor of five or more (Fischer and Morgenstern, 2005), the approximation error introduced by Equation 2 can be considered to be acceptable. Moreover, the purpose of this model is not to make exact predictions of future costs and carbon prices: the purpose is to have an internally consistent model that allows us to compare, in a systematic way, different scenarios of the global carbon market.



**Figure 2:** The diagram shows the fit of the model to the marginal abatement cost data. Although Equation 2 is a very basic formula, the graph shows that it can accurately reproduce the data. In order not to disclose sensitive data, it was decided not to include the cost values.

It is important to note that more accurate fits are possible. Here, however, we have decided to use a very simple function. There are a number of advantages to this. First, it is convenient and concise to specify each marginal abatement cost curve by a single parameter, i.e.  $\rho_i$ . Second this particular function allows us to obtain explicit formulae for all the economic quantities of interest, for example the equilibrium carbon price or the incremental costs incurred by countries. On the other hand, for example, a function of the form  $p = \alpha q + \rho q^2$  would lead to implicit equations<sup>7</sup>. This is not an improvement: a small gain in precision comes at a significant loss in flexibility as the solutions would need to be obtained numerically. It is important to stress the fact that the main purpose of this model is to retain the essence of the problem while getting rid of as much unnecessary complexity as possible. If we are interested in more numerical precision, we can run the full numerical model (GLOCAF). On the other hand, this analytical version of the model retains all the properties of the numerical model but gains in flexibility, while producing results that are very accurate. Moreover, the key strength of the model is in the insights it offers into the problem.

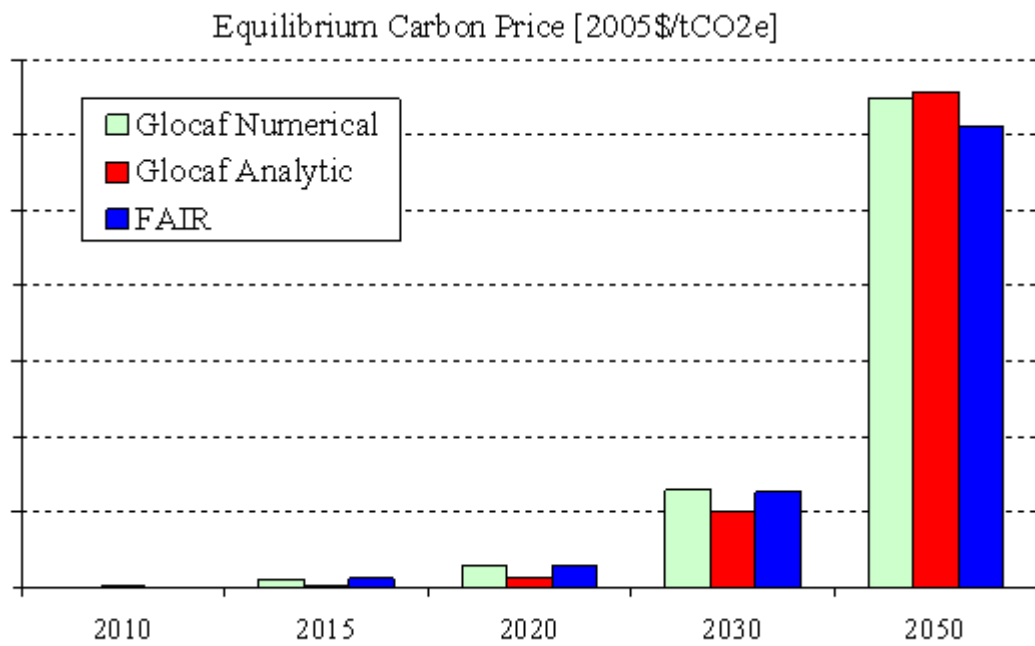
## II.1 Numerical validation

This section compares the results from the analytical model with those obtained from two state-of-the-art numerical models: GLOCAF and FAIR.

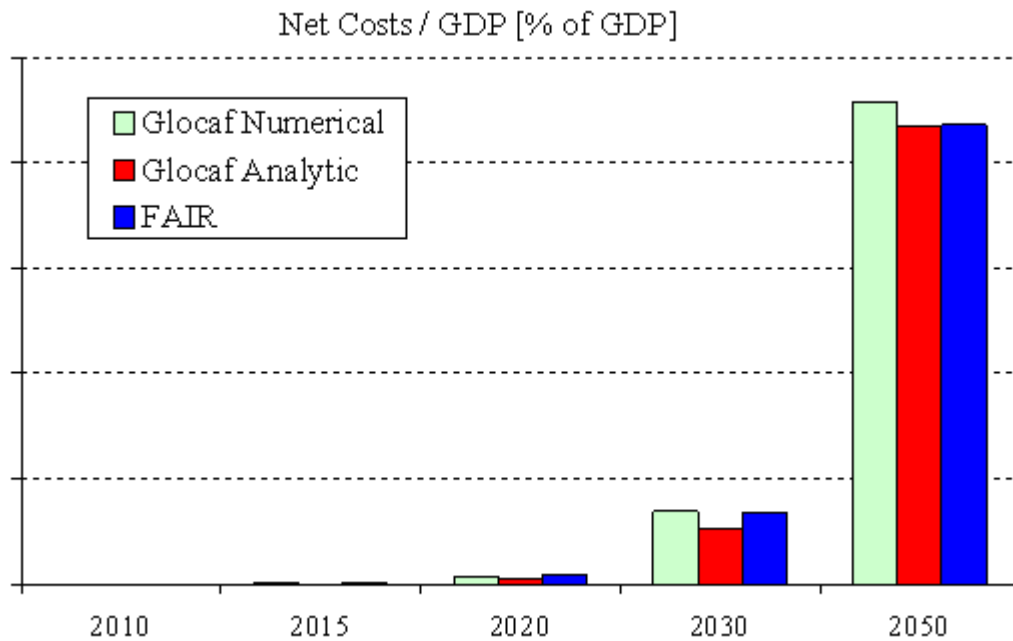
Figure 3 and Figure 4 show that the results from the simple analytic model are in agreement with the numerical models, and that the analytic model is therefore validated. The scenarios presented correspond to free trade. In this example, the emissions pathway leads to a stabilisation concentration of 550ppm and the burden is shared according to Contraction and Convergence.

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<sup>7</sup> An example of implicit equation is  $x = e^x$ . An exact solution to this equation does not exist, and the results need to be approximated numerically. In practise this complicates the implementation of the model: for example it would not be possible to set up a simple spreadsheet, and a numerical iteration would need to be programmed.

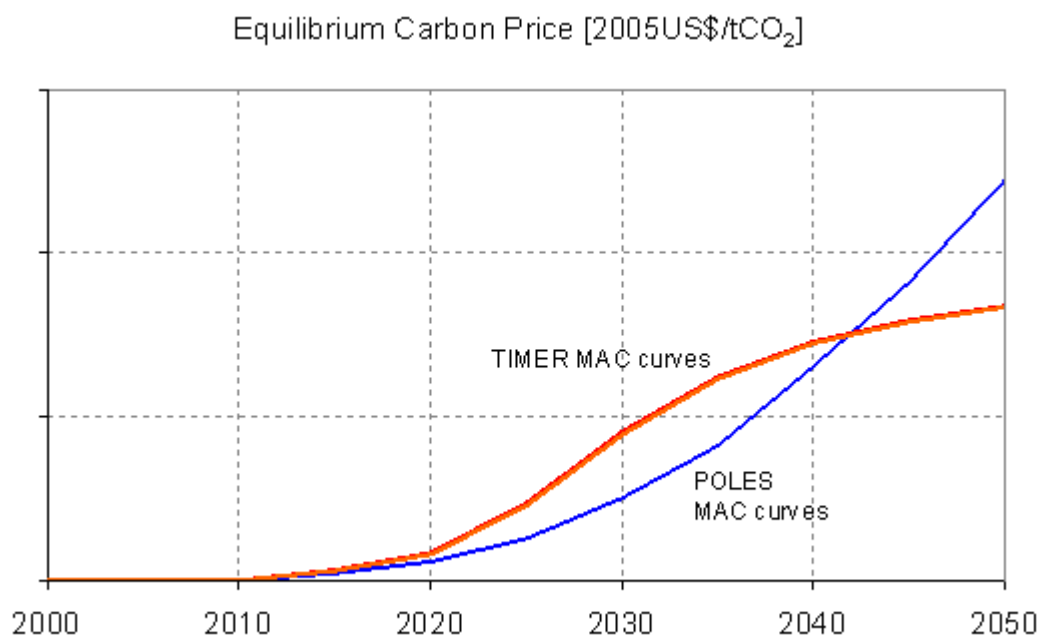


**Figure 3:** The diagram compares the equilibrium carbon prices from the present analytic model with those from GLOCAF and FAIR. In order not to disclose sensitive results, it was decided not to present the actual values of the carbon prices.



**Figure 4:** The diagram compares the global costs as a function of global GDP from the present analytic model with those from GLOCAF and FAIR. In order not to disclose sensitive results, it was decided not to present the actual values of the net costs.

The accuracy of the simple model is particularly evident when we compare the results produced by FAIR based on two different sets of marginal abatement costs, POLES and TIMER. Figure 5 compares two identical scenarios produced by FAIR. The only difference is given by the underlying marginal abatement costs for energy CO<sub>2</sub>: one uses POLES and the other uses TIMER data. Clearly the results are significantly different. In other words, since currently available marginal abatement cost curves from different sources can lead to differences in outputs by a factor of five or more, a small difference between analytic model and the numerical models is negligible.



**Figure 5: Two identical scenarios run by FAIR using POLES and TIMER MAC curves, respectively. In order not to disclose sensitive results, it was decided not to present the actual values of the carbon prices.**